Lyapunov exponents in 1d disordered system with long-range memory

Alexander Iomin
Department of Physics, Technion, Haifa, 32000, Israel
(Dated: February 19, 2009)

The Lyapunov exponents for Anderson localization are studied in a one dimensional disordered system. A random Gaussian potential with the power law decay $\sim 1/|x|^q$ of the correlation function is considered. The exponential growth of the moments of the eigenfunctions and their derivative is obtained. Positive Lyapunov exponents, which determine the asymptotic growth rate are found.

PACS numbers: 72.15.Rn, 05.40.-a

In this paper we consider Anderson localization [1, 2] in a one dimensional disordered system with a long-range memory. The recent realization of disordered systems by using ultra cold atoms [3, 4] in optical lattices and microwave realization of the Hofstadter butterfly [5] show that the random potential in the experiments are highly correlated. The increased interest in the problem of Anderson localization in random potentials with long-range correlations is also relevant to studies of the metal-insulator transition [6, 7].

Anderson localization in a one dimensional disordered system is described in the framework of the eigenvalue problem

$$\epsilon\phi(x) = -\frac{d^2}{dx^2}\phi(x) - V(x)\phi(x)\,, \qquad (1)$$

with a Gaussian random potential $V(x)$. The long-range memory of the disorder is modelled by the two point correlation function $\mathcal{C}(x)$ with the power law decay at the large scale

$$\langle V(x')V(x)\rangle = \mathcal{C}_q(x-x') = \frac{C_q}{|x-x'|^q}\,, \qquad (2)$$

where $q > 0$. It has been shown by various techniques that all eigenfunctions are localized for correlated potentials with the correlation decay rate $0 < q < 1$ [7, 8]. Spectral properties of the random operator of Eq. (1) (and its discrete counterpart) were studied [9, 10, 11]. Due to the physical interpretation, see discussion in Ref. [6], one of the main results is the absence of the absolutely continuous spectra for the random Schrödinger operator (1) with the correlation properties due to Eq. (2). This means that the eigenfunctions $\phi(x)$ are localized, and investigation of Lyapunov exponents is a serious task related to localization of the eigenfunctions.

The Lyapunov exponents are important in spectral theory, since they govern the asymptotic behavior of the wave functions. They are defined on the asymptotic behavior of the averaged envelope $\gamma_s(\epsilon) = \lim_{x\to\infty}\frac{\langle\ln\phi^2(x)\rangle}{2x}$. It was shown by rigorous analysis that the positive Lyapunov exponents are absent for the absolutely continuous spectrum, while the positiveness of the Lyapunov exponents ensures that the spectrum is pure point [11, 12].

In this paper, we calculate $\langle\phi^2(x)\rangle$ of solutions of Eq. (1) for a certain energy ϵ, with given boundary conditions at some point, for example $\phi(x=0)$ and $\phi'(x=0)$, where prime means the derivative with respect to x. Since the distribution of random potentials is translationally invariant, it is independent of the choice of the initial point as $x = 0$. It will be shown that this quantity grows exponentially with the rate $\gamma(\epsilon) = \lim_{x\to\infty}\frac{\ln\langle\phi^2(x)\rangle}{x} > 0$. Note that it is different from γ_s, which supposes a knowledge of all the even moments [13, 14, 15, 16].

We develop a general procedure which is suitable for calculation of all moments of the wave function and its first derivative. To this end the Schrödinger equation (1) is considered as the Langevin equation and the x coordinate as a formal time. For the δ correlated process it can be easily mapped on the Fokker-Planck (diffusion) equation for the probability distribution function $\mathcal{P}(\phi,\phi')$ [12, 17]. Unlike this, the two point correlation function (2), which corresponds to the stationary process, leads to additional integration over the formal "time" with a memory kernel. The method of consideration enables one to observe the exponential growth of $\langle\phi^2(x)\rangle$ with the Lyapunov exponent $\gamma(\epsilon) > 0$.

Since the Schrödinger equation (1) is a linear stochastic equation, equations for the $2n$ moments of the type

$$M_{k,l}(x) = \langle[\phi(x)]^k[\phi'(x)]^l\rangle\,,\ \ k+l = 2n,\ \ k,l = 0,1,2,\dots\,, \qquad (3)$$

can be obtained in the closed form. To this end we rewrite Eq. (1) in the form of the Langevin equation. The x coordinate is considered as a formal time on the half axis $x \equiv \tau,\ \ \tau \in [0,\infty)$ and the new dynamical variables $u(\tau) = \phi(x)$, $v(\tau) = \dot{u} = \phi'(x)$ are defined. In the new variables the Langevin equation reads

$$\dot{u} = v\,,\quad \dot{v} = -[\epsilon + V(\tau)]u\,, \qquad (4)$$

where $V(\tau)$ is now the long-range correlated noise

$$\mathcal{C}_\alpha(\tau) = \frac{C_\alpha}{\tau^{1+\alpha}}\,. \qquad (5)$$

It is convenient to set $q = 1+\alpha$ and $C_q \equiv C_\alpha$. In the new variables the expectation values of Eq. (3) are now $M_{k,l}(\tau) = \langle u^k v^l\rangle$. Solutions of Eq. (4) are obtained as functionals

$$v(t) = -\int_0^t [\epsilon + V(\tau)]u(\tau)d\tau\,,\quad u(t) = \int_0^t v(\tau)d\tau\,. \quad (6)$$

Following [15] we obtain a temporal equation for the moments from the Langevin equation (4) and its solutions (6). Differentiating $M_{k.l}(\tau)$ with respect to τ, we obtain

$$\dot{M}_{k,l} = kM_{k-1,l+1} - l\epsilon M_{k+1,l-1} - l\langle V(t)u^{k+1}v^{l-1}\rangle\,. \quad (7)$$

The application of the Furutsu-Novikov formula [18] to the last term in Eq. (7) yields

$$\begin{aligned}\langle V(t)\mathcal{F}[V(t)]\rangle &= \tfrac{1}{2}\int_0^t d\tau'\langle V(t)V(\tau')\rangle\left\langle \tfrac{\delta\mathcal{F}[V(\tau)]}{\delta V(\tau')}\right\rangle \\ &= -\tfrac{1}{2}(l-1)\int_0^t \mathcal{C}_\alpha(t-\tau)M_{k+2,l-2}(\tau)d\tau\,. \end{aligned} \quad (8)$$

Here the solution of Eq. (6) is used to obtain the functional derivative of the functional $\mathcal{F}[V(\tau)] = u^{k+1}v^{l-1}$. Substituting the solution of Eq. (8) in Eq. (7), we obtain that the temporal behavior of the moments is described by the fractional–differential equation

$$\dot{M}_{k,l} = kM_{k-1,l+1} - l\epsilon M_{k+1,l-1} + \frac{1}{2}l(l-1)D_t^\alpha M_{k+2,l-2}\,, \quad (9)$$

where the convolution integral in Eq. (8) is the fractional derivative $D_t^\alpha f(t)$

$$D_t^\alpha f(t) = C_\alpha \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1+\alpha}}\,. \quad (10)$$

Here the correlation function $\mathcal{C}_\alpha(t)$ defines the memory kernel, or the causal function. Eqs. (9) and (10) are relevant to the fractional Fokker-Planck equations which describe a variety of physical processes related to fractional diffusion [19, 20, 21]. An important technique for the treatment of the fractional equation is the Laplace transform. It is worth stressing that both analytical properties of this fractional integration and the Laplace transform depend on α.

For $-1 < \alpha < 0$ Eq. (9) is readily solved by means of the Laplace transform. Defining $\hat{\mathcal{L}}[M_{k,l}(t)] = \tilde{M}_{k,l}(s)$, one obtains from Eq. (10) $\hat{\mathcal{L}}[D_t^\alpha M_{k,l}(t)] = C_\alpha\Gamma(-\alpha)s^\alpha\tilde{M}_{k,l}(s)$, where $\Gamma(\alpha)$ is the gamma function. For simplicity, disregarding the sign of the correlation function (5), we set $C_\alpha = 2/\Gamma(-\alpha)$. Then, we introduce $2n+1$-dimensional vectors $\mathbf{M}_n(t) = \Big(M_{2n,0}, M_{2n-1,1}, \dots, M_{1,2n-1}, M_{0,2n}\Big)$ in the "time" space and $\tilde{\mathbf{M}}_n(s) = \hat{\mathcal{L}}[\mathbf{M}_n(t)]$ in the Laplace space, correspondingly. Then the solution of Eq. (9) is the Laplace inversion of the following vector

$$\tilde{\mathbf{M}}_n(s) = \frac{1}{s - A_n(s)}\mathbf{M}_n(0)\,, \quad (11)$$

where $(2n+1)\times(2n+1)$ matrix $A_n(s)$ consists of coefficients from the matrix equation (9). In the limit $s \to 0$ the disorder term of order of $s^\alpha \to \infty$ is dominant, and the maximal eigenvalues of A_n can be evaluated at the energy $\epsilon \approx 0$. Following Ref. [16], it can be proven that for $\epsilon = 0$ the maximal eigenvalues of A_n behaves for large n as $\Omega(s) \approx s^{\alpha/3}(2n)^{4/3}$. Expanding the initial condition $M_n(0)$ over the eigenfunctions of A_n, we obtain that the maximal growth of the nth moment is

$$M_n(t) = \hat{\mathcal{L}}^{-1}\Big[\frac{s^{-\alpha/3}}{s^{1-\alpha/3} - (2n)^{4/3}}\Big]M_\Omega(0)\,. \quad (12)$$

The inverse Laplace transform is the definition of the Mittag-Leffler function [22]: $E_{1-\alpha/3}\Big(\frac{3}{4}(2n)^{4/3}t^{1-\alpha/3}\Big)$. Asymptotic behavior of the Mittag-Leffler function for $t \to \infty$ is determined by the exponential function $\exp\Big[(2n)^{4/(3-\alpha)}t\Big]$. Therefore the exponential growth of the nth moment is due to the Lyapunov exponent

$$\gamma(0) \sim (2n)^{4/(3-\alpha)} \quad (13)$$

for $-1 < \alpha < 0$.

For $\alpha > 0$ the fractional integral diverges. To overcome this obstacle, one considers the causal function as a generalized function, and a suitable regularization procedure can be carried out see e.g., [20, 21]. Let $N-1 < \alpha < N$, where $N \ge 1$ is an integer. Again, using the composition rule, one obtains the Riemann-Liouville fractional integral (10) in the regularized form

$$\begin{aligned} D_t^\alpha f(t) &= D_t^N D_t^{\alpha-N} f(t) \equiv D_{RL}^\alpha f(t) \\ &= \frac{1}{\Gamma(N-\alpha)}\frac{d^N}{dt^N}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1+\alpha-N}}\,. \end{aligned} \quad (14)$$

Thus Eq. (9) reads

$$\dot{M}_{k,l} = kM_{k+1,l-1} - l\epsilon M_{k+1,l-1} + l(l-1)D_{RL}^\alpha M_{k+2,l-2}\,. \quad (15)$$

This fractional equation of the order of α must be equipped with $N-1$ quasi initial conditions: in addition to the initial conditions $M_{k,l}(0)$, one has to know $N-1$ fractional derivative of $M_{k,l}(\tau)$ at $\tau = 0$. Application of the Laplace transform to the fractional derivative yields [20]

$$\hat{\mathcal{L}}[D_{RL}^\alpha M_{k,l}(t)] = s^\alpha\tilde{M}_{k,l}(s) - \sum_{p=0}^{N-1} s^p D_{RL}^{\alpha-1-p}M_{k,l}(t)\Big|_{t=0}\,. \quad (16)$$

In the asymptotic limit $s \to 0$ we obtain that the solution of Eq. (15) is approximated by the inverse Laplace transform of the vector

$$\tilde{\mathbf{M}}_n(s) \approx \frac{1}{s - A_n(s)}[I_n - B_n D_{RL}^{\alpha-1}]\mathbf{M}_n(0)\,, \quad (17)$$

where I_n is an unit matrix and matrix B_n consists of the off diagonal elements which produce $M_{k+2,l-2}$ terms in Eq. (15).

Since we are seeking the maximal growth rate of the solution of Eq. (15) and the initial condition are not important for this growth, we choose the initial condition as the eigenvector of the maximal eigenvalue of the matrix A_n. In what follows we consider a temporal behavior of the second moments, described by 3×3 matrix

$A_1(s)$. The eigenvalues of the matrix are roots of a cubic equation [23]. The growth rate is determined by the eigenvalue with the largest real part that will be denoted by Ω. Taking the initial condition in Eq. (17 as the eigenfunction of Ω, namely $-\mathbf{M}_\Omega(0)$, we obtain that the dynamics of the second moments is due to the Laplace inversion

$$\mathbf{M}_1(t) \propto \hat{\mathcal{L}}^{-1}\left[(\Omega(s) - s)^{-1}\right]. \quad (18)$$

For small s the eigenvalues $\Omega(s)$ correspond to a "weak" disorder in the Laplace space. Therefore the high energy limit is valid $\Omega \approx s^\alpha/\epsilon$, where $\epsilon \gg s^\alpha$ [12, 23]. Substituting this eigenvalue in Eq. (18) and expanding the denominator we have for the integrand $\sum_{n=0}^{\infty} \epsilon^{n-1}(1/s)^{(\alpha-1)n+\alpha}$. Carrying out the Laplace inversion, we obtain the solution in the form of another definition of the Mittag-Leffler function (see e.g., [20, 24]) $E_{\alpha-1,\alpha}(\epsilon t^{\alpha-1}) = \sum_{n=0}^{\infty} \frac{(\epsilon t^{\alpha-1})^n}{\Gamma(n\alpha-n+\alpha)}$. Therefore

$$\mathbf{M}_1(t) \propto \epsilon t^{\alpha-1} E_{\alpha-1,\alpha}(\epsilon t^{\alpha-1}). \quad (19)$$

Since the argument of the Mittag-Leffler function is positive $\epsilon t^{\alpha-1} > 0$, then the asymptotic behavior is approximately $E_{\beta,\delta}(z) \approx z^{(1-\delta)/\beta}\exp(z^{1/\beta})$ for $z \to \infty$ for all values δ [22, 24]. Therefore, when $\epsilon t^{\alpha-1} \to \infty$ the exponential growth of the second moment

$$\mathbf{M}_1(t) \propto \exp[\gamma(\epsilon)t] \quad (20)$$

is approximated by the Lyapunov exponent

$$\gamma(\epsilon) \sim \epsilon^{1/(\alpha-1)}. \quad (21)$$

Another way to obtain the Lyapunov exponents avoiding the difficulties related to the $N-1$ quasi initial conditions in Eqs. (15) and (16) is to discard the causality principle and extend the consideration of the random process on the entire x axis $x \in (-\infty, +\infty)$. For this formal consideration, the Furutsu-Novikov formula in Eq. (8) reads

$$\begin{aligned} &-(l-1)C_\alpha \int_{-\infty}^{x} \frac{M_{k+2,l-2}(y)}{(x-y)^{1+\alpha}}dy \quad \text{for } x>0, \\ &-(l-1)C_\alpha \int_{x}^{\infty} \frac{M_{k+2,l-2}(y)}{(y-x)^{1+\alpha}}dy \quad \text{for } x<0. \end{aligned} \quad (22)$$

Setting again $C_\alpha = \Gamma(-\alpha)$, we obtain that Eq. (22) is the definition of the Riesz/Weyl fractional derivative $\mathcal{W}_x^\alpha$ see e.g., [20, 21, 24]. Therefore, Eq. (9) now reads

$$\frac{d}{dx}M_{k,l} = kM_{k-1,l+1} + l\epsilon M_{k+1,l-1} + l(l-1)\mathcal{W}_x^\alpha M_{k+2,l-2}. \quad (23)$$

A specific property that we use is the fractional differentiation of an exponential $\mathcal{W}_x^\alpha \exp(\gamma x) = \gamma^\alpha \exp(\gamma x)$. Substituting this in Eq. (7), one seeks the solution for the maximal moment growth $M_{k,l}(x) = \exp(\pm\gamma x)M_{k,l}(x=0)$, where plus stays for $x>0$ and minus for $x<0$, respectively. One readily checks that the both cases yield the same algebraic equation

$$\gamma\mathbf{M}_n = A_n(\gamma)\mathbf{M}_n, \quad (24)$$

where the moment vector $\mathbf{M}_n$ is defined above and the matrix $A_n(\gamma)$ is defined from Eq. (23). Therefore, $\Omega(\gamma) = \gamma^\alpha/\epsilon$, where conditions $\gamma \ll \epsilon$ and $\gamma^\alpha \ll \epsilon$ are used. Solutions of Eq. (23) for $\gamma(\epsilon)$ coincide exactly with the ones obtained in Eqs. (13) and (21) for all values of α.

This solution for γ also yields conditions of validity of the solution (21) for different values of energy ϵ. Indeed, for $0<\alpha<1$ Eqs. (20), (21) and (24) describe an exponential growth for asymptotically large energies $\epsilon \gg 1$, since, in this case, $\gamma(\epsilon) \ll \epsilon$ when $\epsilon \gg 1$. On the contrary, when $\alpha > 1$ the solution of Eq. (21) is valid for $\epsilon \ll 1$. This follows from the condition $\gamma^\alpha \ll \epsilon$. Note that for large negative values of the energy $\Omega \sim 2\sqrt{|\epsilon|}$, what corresponds to a simple pole in Eq. (18), and this is just the Lyapunov exponent $\gamma(\epsilon) \sim 2\sqrt{|\epsilon|}$.

In conclusion, we studied the Lyapunov exponents for Anderson localization in a one-dimensional disordered system with a long-range memory. The averaged behavior of the second moment of the eigenfunction is calculated, and its asymptotic exponential growth for $|x| \to \infty$ is determined by the Lyapunov exponents for different values of the energy ϵ. The main result of the study is the existence of the positive Lyapunov exponents $\gamma(\epsilon) > 0$ for the rate $q = 1+\alpha > 0$ of the power law decay of the correlation function. It is relevant to the exponential localization of the eigenfunctions of the random Schrödinger operator of Eq. (1).

This work was supported by the Israel Science Foundation. I thank S. Fishman for very informative and instructive discussions.

[1] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[3] L. Sanchez-Palencia et al., Phys. Rev. Let. **98**, 210401 (2007); J. Billy et al., Nature **453**, 891 (2008).
[4] G. Roati et al., Nature **453**, 895 (2008).
[5] U. Kuhl et al., Appl. Phys. Let. **77**, 633 (2000); U. Kuhl and H.-J. Stöckmann, Phys. Rev. lett. **80**,3232 (1998).
[6] A.M. Garcia-Garcia and E. Cuevas, Absence of localization in one-dimensional disordered systems, cond-mat.0808.3757.
[7] F.A.B.F. de Moura and M.L. Lyra, Phys. Rev. Let. **81**,

3735 (1998).
[8] F.A.B.F. de Moura and M.L. Lyra, Physica A **266**, 465 (1999); S. Russ, et al., Physica A **266**, 492 (1999).
[9] S. Kotani, Proc.Kyoto Stoch. Com. (1982); B. Simon, Comm. Math. Phys. **89**, 227 (1983).
[10] S. Kotani and B. Simon, Comm. Math. Phys. **112**, 103 (1987).
[11] L.A. Pastur and A.L. Figotin, *Spectra of Random and Almost-Periodic Operators* (Springer, Berlin, 1992).
[12] I.M. Lifshits, S.A. Gredeskul, and L.A. Pastur, *Introduction to the theory of disordered systems* (Wiley-Interscience, New York, 1988).
[13] K. Mallick and P. Marcq, Phys. Rev. E **66**, 041113 (2002).
[14] H. Schomerus and M. Titov, Phys. Rev. E **66**, 066207 (2002).
[15] R. Zilmer and A. Pikovsky, Phys. Rev. E **67** 061117 (2003).
[16] S. Fishman, A. Iomin, and K. Mallick, Phys. Rev. E **78**, 066605 (2008).
[17] B.I. Halperin, Phys. Rev. **139**A, 104 (1965).
[18] V.I. Kliatskin, *Stochastic equations and waves in randomly inhomogeneous media* (Nauka, Moskva, 1980) (in Russian).
[19] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[20] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[21] G.M. Zaslavsky, Phys . Rep. **371**, 461 (2002).
[22] H. Bateman and A. Erdélyi *Higher Transcendental functions* (Mc Graw-Hill, New York, 1955), V. 3.
[23] A. Iomin and S. Fishman Phys. Rev. E **76**, 056607 (2007).
[24] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).